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Note on Howard's semicircle theorem

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For a steady plane parallel flow of an inviscid incompressible fluid of variable density under gravity, it is shown that the complex wave velocity for any unstable mode lies in a semi-ellipse whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends on the stratification.

1. Introduction

For a steady plane parallel flow of an inviscid incompressible fluid of variable density under gravity, Howard (1961, henceforth referred to as I) proved that the complex wave velocity c for any unstable mode must lie inside the semicircle in the upper half-plane which has the range of the basic velocity for its diameter. It was observed that the same result holds for homogeneous fluids. The physical nature of the problem strongly suggested that for an unstable mode c must lie in a region dependent on the density stratification. This question has remained unanswered for all these years. It is shown in this note that for any unstable mode c must lie in a certain semi-ellipse, rather than a semicircle, whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends on the stratification.

2. Howard's result

To facilitate reference to I, the same notations will be used here. The basic velocity is $U(y)$, the density field $\rho(y)$ and g is the acceleration of gravity. β denotes $-\rho'/\rho$ and the stream-function perturbation is

$$(U-c)\,F(y)\exp\left[ik(x-ct)\right].$$

It is assumed that the density stratification is statically stable ($\beta \geqslant 0$). The boundary conditions are that F vanishes on $y = y_1$ and $y = y_2$ (rigid walls).

The flow is unstable if the linearized equations of motion and the boundary conditions have non-trivial solutions with $\operatorname{Im} c > 0$. Set $c = c_r + ic_i$ and assume that F is such an unstable solution. Then, since $c_i > 0$, a proper meaning can be attached to the transformation

$$G = (U-c)^{\frac{1}{2}}\,F. \tag{1}$$

The linearized equation of motion is now transformed in terms of G along with the boundary conditions $G(y_1) = G(y_2) = 0$. If this equation is multiplied by the complex conjugate of G and integrated over (y_1, y_2), then its imaginary part implies that

$$\int \rho(|G'|^2+k^2|G|^2) = \int \left(\frac{U'^2}{4}-\beta g\right)\frac{\rho|G|^2}{|U-c|^2}. \tag{2}$$

This is equation (2.4) of I.

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If the original linear equation of motion for F is multiplied by its complex conjugate and integrated over (y_1, y_2), then the real and imaginary parts together with certain ingenious manipulations introduced by Howard imply the inequality

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right] \int \rho Q + \int \rho g \beta |F|^2 \leqslant 0, \tag{3}$$

where
$$Q = |F'|^2 + k^2|F|^2, \quad a \leqslant U(y) \leqslant b. \qquad (4), (5)$$

This is the main (unnumbered) inequality in I.

Howard drops the last term, which is positive, in the inequality (3) to establish the semicircle theorem. However, it is to be noted that the dropped term is the only term in the inequality which explicitly incorporates the effect of stratification. A positive, lower and relevant estimate of the dropped term paves the way for the proof of a semi-ellipse theorem.

3. Semi-ellipse theorem

First, an estimate of the dropped term will be obtained. Let us differentiate (1) to get

$$|G'|^2 \geqslant |U-c|\,|F'^2| + \frac{U'^2|F|^2}{4|U-c|} - |U'|\,|F|\,|F'|.$$

Let
$$B^2 = \int \frac{\rho U'^2 |F|^2}{4|U-c|}, \quad E^2 = \int \rho |U-c|\, Q, \quad J_0 = \left[\frac{g\beta(y)}{U'^2}\right]_{\min},$$

where $J(y)$ denotes the local Richardson number. Use of this inequality and (1) in (2) implies

$$(1-4J_0)B^2 \geqslant E^2 + B^2 - \int \rho |U'|\,|F|\,|F'|. \tag{6}$$

It is to be noted that, under the assumption that $c_i > 0$, Miles' necessary condition for instability (i.e. $J_0 < \frac{1}{4}$) has explicitly been used.

Now the Schwarz inequality can be used to get

$$\begin{aligned}\int \rho |U'|\,|F|\,|F'| &\leqslant \left[\int \frac{\rho U'^2|F|^2}{|U-c|} \int \rho |U-c|\,|F'|^2\right]^{\frac{1}{2}} \\ &\leqslant \left[\int \frac{\rho U'^2|F|^2}{|U-c|} \int \rho|U-c|(|F'|^2 + k^2|F|^2)\right]^{\frac{1}{2}} \\ &= 2BE.\end{aligned}$$

This helps to reduce (6) to the inequality

$$1-(1-4J_0)^{\frac{1}{2}} \leqslant E/B \leqslant 1+(1-4J_0)^{\frac{1}{2}}.$$

Then for $c_i > 0$ the right-hand inequality can be rewritten as

$$\begin{aligned}c_i \int \rho Q \leqslant \int \rho |U-c|\, Q &= E^2 \\ &\leqslant [1+(1-4J_0)^{\frac{1}{2}}]^2 B^2 \\ &= \tfrac{1}{4}[1+(1-4J_0)^{\frac{1}{2}}]^2 \int \frac{\rho U'^2|F|^2}{|U-c|} \\ &\leqslant \frac{1}{4c_i}[1+(1-4J_0)^{\frac{1}{2}}]^2 \int \rho U'^2 |F|^2.\end{aligned}$$

Therefore

$$\int \rho g \beta |F|^2 \geqslant J_0 \int \rho U'^2 |F|^2 \geqslant \frac{4c_i^2 J_0}{[1+(1-4J_0)^{\frac{1}{2}}]^2} \int \rho Q.$$

We have thus established the following lemma.

LEMMA. For an unstable mode, we must have

$$\int \rho g \beta |F|^2 \geqslant \frac{4J_0 c_i^2}{[1+(1-4J_0)^{\frac{1}{2}}]^2} \int \rho Q. \tag{7}$$

THEOREM The complex wave velocity c for an unstable mode must lie inside the semi-ellipse in the upper half-plane which has the range of U for its major axis, and $\{\frac{1}{2}[1+(1-4J_0)^{\frac{1}{2}}]\}^{\frac{1}{2}}$ times the range of U for its minor axis.

Let us use the estimate given by the lemma of the dropped term in the inequality (3) to get

$$\left[\{c_r - \tfrac{1}{2}(b+a)\}^2 + \left\{1 + \frac{4J_0}{[1+(1-4J_0)^{\frac{1}{2}}]^2}\right\} c_i^2 - \tfrac{1}{4}(b-a)^2\right] \int \rho Q \leqslant 0.$$

This implies that

$$[c_r - \tfrac{1}{2}(a+b)]^2 + \frac{2c_i^2}{1+(1-4J_0)^{\frac{1}{2}}} \leqslant \tfrac{1}{4}(b-a)^2. \tag{8}$$

In particular, for a homogeneous fluid for which $J_0 = 0$, the semi-ellipse theorem reduces to Howard's semicircle theorem.

4. Concluding remarks

A necessary and sufficient condition for instability is a rare feature in the theory of hydrodynamic stability. The present problem is no exception. In the absence of a sufficient condition for instability, different necessary conditions are helpful in understanding a problem better. The present problem is one where quite a number of necessary conditions are available (Howard & Drazin 1966). Howard's semicircle theorem provides one such necessary condition. There have been attempts to improve upon Howard's necessary condition to take density stratification into consideration. Banerjee & Jain (1972) and Banerjee, Gupta & Gupta (1974) were able to reduce Howard's semicircle, however these results are conditional on the velocity distribution. On the one hand, the semi-ellipse theorem established in this note is not a conditional result and on the other hand it incorporates the effect of stratification in a more natural fashion, in the sense that the minor axis depends on J_0.

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